SOLITARY WAVES IN A THERMO-ELASTIC MEDIUM

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Using a one-dimensional model, we study conditions for the appearance of solitary waves in a thermo-elastic medium as a function of the nonlinear nature of heat conduction, temperature, and thermal expansion coefficient. Two conditions for the appearance of solitary waves are examined, one of which has a physical meaning and therefore confirms the fact of the existence of solitons in a thermo-elastic medium. Geometric characteristics of solitons, their speed, amplitude, and width, are obtained.

1. Statement of the Problem. The propagation of one-dimensional waves in a thermo-elastic medium in the absence of sources of heat generation and absorption may be described by the following system of equations [1, 2]:

$$\frac{\partial^2\sigma}{\partial x^2}-\frac{1}{f^2}\frac{\partial^2\sigma}{\partial t^2}=\rho\frac{\partial^2 T}{\partial t^2}, \tag{1}$$

$$\frac{\partial^2\theta}{\partial x^2}-\frac{1}{\varkappa}\frac{\partial\theta}{\partial t}-\frac{1}{\lambda_0}T\frac{\partial}{\partial t}(\sigma+9KT)=0. \tag{2}$$

Equation (2) is the equation of heat conduction for media, based on Fourier's law.

Based on these equations, we consider the following problem. Find the form of the nonlinear dependence $\theta = \theta(\sigma)$, $\alpha = \alpha(\sigma)$ for which equations (1), (2) admit a solution of solitary wave type. In order for the desired solution of equations (1), (2) to have the nature of a solitary wave, the stress σ must satisfy the equation [3]:

$$\frac{\partial^2\sigma}{\partial x^2}-[k^2+v(x,\ t)]=0. \tag{3}$$

The condition k^2 = const is equivalent to the following equations:

$$\frac{\partial\sigma}{\partial t}=-u\frac{\partial^3\sigma}{\partial x^3}+6v\frac{\partial\sigma}{\partial x}+3\sigma\frac{\partial v}{\partial x}-\lambda_3\sigma, \tag{4}$$

$$\frac{\partial v}{\partial t}-6v\frac{\partial v}{\partial x}+\frac{\partial^3 v}{\partial x^3}=0, \tag{5}$$

where k^2 and λ_3 are constants; v is a solution of equation (5). Since equations (1)-(5) do not contain the variables x and t explicitly, they admit self-similar solutions for which equations (1)-(5) can be transformed as follows:

$$2v+c-4k^2=\left(\frac{\lambda_3}{g}+\frac{dv}{d\sigma}\right)\sigma, \tag{6}$$

$$\frac{dv}{d\sigma}g=(2v^3+cv^2+dv+b)^{1/2}, \tag{7}$$

$$h=-cg, \tag{8}$$

$$\rho\frac{dT}{d\sigma}=\frac{1}{c^2}-\frac{1}{f^2}-\frac{\rho}{mg}, \tag{9}$$

$$\frac{d\theta}{d\sigma}+\frac{c}{\varkappa g}\theta=\frac{n}{g}-\frac{1}{g}\left[\frac{c}{\lambda_0}\int_0^\sigma T\,d\tau+\frac{9c}{2\lambda_0}KT^2\right], \tag{10}$$

where b, c, d, m, and n are constants of integration, and

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$$g(\sigma)=\frac{\partial\sigma}{\partial x}, \quad h(\sigma)=\frac{\partial\sigma}{\partial t}. \tag{11}$$

Equations (6)-(10) constitute a complete set of equations describing the propagation of solitary waves in a thermo-elastic medium.

2. Finding a Solution of Equation (9). For $c^2 = f^2 = E/\rho$ it follows from equation (9) that

$$T=-\frac{1}{m}\int_0^\sigma \frac{d\tau}{g(\tau)}. \tag{12}$$

We introduce a new independent function s by means of the equation

$$s=\frac{1}{2}\left(v+\frac{c}{6}\right). \tag{13}$$

On the basis of equation (7) we have

$$\int_{e_3}^{s} \frac{dy}{(4y^3-g_2y-g_3)^{1/2}}=\int_p^\sigma \frac{d\tau}{g(\tau)}, \tag{14}$$

where e_3 is the smallest real root of the equation

$$\begin{gathered} 4y^3-g_2y-g_3=0; \\ g_2=\frac{c^2}{12}-\frac{d}{2};\quad g_3=\frac{cd}{24}-\frac{c^3}{6^3}-\frac{b}{4}; \end{gathered} \tag{15}$$

p is a constant which satisfies the equation $y(p) = e_3$. Using relations (12) and (14), we obtain

$$\int_{-\infty}^{s} \frac{dy}{(4y^3-g_2y-g_3)^{1/2}}=q-mT+\omega_3, \tag{16}$$

where

$$q=-\int_0^p \frac{d\tau}{g(\tau)}, \quad \omega_3=\int_{-\infty}^{e_3} \frac{dy}{(4y^3-g_2y-g_3)^{1/2}}. \tag{17}$$

From relation (16) we have

$$s=\wp(z+\omega_3), \tag{18}$$

where $\wp(z+\omega_3)$ is the elliptic function of Weierstrass [4];

$$z=q-mT. \tag{19}$$

Using relation (13), we write relation (18) explicitly thus:

$$v=2e_3-\frac{c}{6}+2(e_2-e_3)\,sn^2[(e_1-e_3)^{1/2}z,\ r], \tag{20}$$

where sn(u, r) is the elliptic function of Jacobi [4],

$$r^2=\frac{e_2-e_3}{e_1-e_3}. \tag{21}$$

From equations (6) and (7) it follows that

$$\frac{dv}{2v+c-4k^2}+\frac{\lambda_3 dv}{(2v+c-4k^2)(2v^3+cv^2+dv+b)^{1/2}}=\frac{d\sigma}{\sigma}. \tag{22}$$

When $2v + c - 4k^2 < 0$, the solution of equation (22) has the form

$$(2v+c-4k^2)\,e^{\lambda_3 I_3}=-4R^2\sigma^2, \tag{23}$$

where R^2 is a constant of integration; I_3 is a Weierstrass elliptic integral of the third kind [4]:

$$I_3=\int \frac{dv}{(2v+c-4k^2)(2v^3+cv^2+dv+b)^{1/2}}. \tag{24}$$

Taking relations (13) and (18) into account, we can put integral (24) in the following form:

$$I_3 = \left[-\frac{1}{2\gamma} + \frac{\delta+\gamma e_3}{\gamma^2\wp'(\varphi)}\xi(\varphi)\right]z + \frac{\delta+\gamma e_3}{2\gamma^2\wp'(\varphi)}\ln\frac{\tilde{\sigma}(z-\varphi)}{\tilde{\sigma}(z+\varphi)}, \tag{25}$$

where

$$\gamma = e_3 + \frac{c}{6} - k^2;\ \delta = \left(e_3 - \frac{c}{6} + k^2\right)e_3 + e_1e_2;\ \wp(\varphi) = -\frac{\delta}{\gamma}; \tag{26}$$

$\xi(u)$, $\tilde{\sigma}(u)$, $\wp'(u)$ are Weierstrass elliptic functions [4]. If we take

$$k^2 - \frac{c}{6} = \frac{e_2(e_1-e_3)^{1/2} + e_3(e_1-e_2)^{1/2}}{(e_1-e_3)^{1/2} + (e_1-e_2)^{1/2}}, \tag{27}$$

then

$$\wp(\varphi) = e_1 + H_1,\ \varphi = \frac{\omega}{2}, \tag{28}$$

where $H_1^2 = (e_1 - e_2)(e_1 - e_3)$ and ω is the real half-period of the Weierstrass elliptic function. In this case it is necessary that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$, since each identical pair e_i, e_j will yield $k^2 - c/b = e_2$, $\wp'(\phi) = \wp'(\omega) = 0$, and relation (25) loses its meaning. In addition, if we take p such that

$$q = \frac{\omega}{2}, \tag{29}$$

then when $\phi = q = \omega/2$, we have

$$I_3 = -\lambda_2\left(\frac{\omega}{2} - mT\right) - \lambda_1 \ln H, \tag{30}$$

where

$$\lambda_2 = \frac{1}{8H_1},\ \lambda_1 = \frac{1}{16H_1[(e_1-e_3)^{1/2} - (e_1-e_2)^{1/2}]}. \tag{31}$$

On the basis of relations (23), (20), (27), and (30) we have

$$\sigma = \frac{1}{R}(e_2-e_3)^{1/2}H_1^{-\frac{1}{2}\lambda_3\lambda_1}\exp\left(-\frac{1}{2}\lambda_3\lambda_2\left(\frac{\omega}{2}-mT\right)\right)\times$$
$$\times\left\{\frac{(e_1-e_3)^{1/2}}{(e_1-e_3)^{1/2}+(e_1-e_2)^{1/2}} - sn^2\left[(e_1-e_3)^{1/2}\left(\frac{\omega}{2}-mT\right),\ r\right]\right\}^{1/2}. \tag{32}$$

3. Finding a Solution of Equation (10). From equation (10) we have

$$\theta = e^{aT}\int_0^\sigma\left\{\frac{n}{g} - \frac{1}{g}\left[\frac{c}{\lambda_0}\int_0^\tau T ds + \frac{9K}{2\lambda_0}T^2\right]\exp(-aT)\right\}d\tau, \tag{33}$$

where $a = mc/\kappa = mEC\varepsilon/\rho\lambda_0$, and σ is calculated from expression (32). From relations (6) and (7) it follows that

$$g = \frac{\sigma}{2v + c - 4k^2}[(2v^3 + cv^2 + dv + b)^{1/2} + \lambda_3]. \tag{34}$$

Introducing relations (23) and (30) into relation (34), we obtain

$$g = -\frac{H_1^{-\lambda_3\lambda_1}}{4R^2\sigma}[(2v^3 + cv^2 + dv + b)^{1/2} + \lambda_3]\exp\left(-\lambda_3\lambda_2\left(\frac{\omega}{2} - mT\right)\right). \tag{35}$$

When $\lambda_3 > 0$, we then have

$$g = -\infty \text{ for } \sigma = 0,\ g < 0\ \forall\sigma. \tag{36}$$

Then by relation (12),

$$T > 0\ \forall\sigma \text{ for } m > 0. \tag{37}$$

Using relations (36) and (37), we find, from relations (10) and (33), that

$$\theta = 0 \text{ for } \sigma = 0;\ \theta > 0,\ \frac{d\theta}{d\sigma} > 0\ \forall\sigma \text{ for } n > 0,\ c > 0. \tag{38}$$

If $\lambda_3 = 0$, then from relations (13), (23), and (35) we have

$$g = -\frac{1}{R^2\sigma}[(s-e_1)(s-e_2)(s-e_3)]^{1/2}, \tag{39}$$

$$s = -R^2\sigma^2 + \frac{c}{12} \quad \text{for } c = 4k^2. \tag{40}$$

Substituting equation (40) into equation (39), we obtain

$$g = -R[(l^2+\sigma^2)(\beta^2-\sigma^2)]^{1/2} < 0, \tag{41}$$

where

$$l^2 = \frac{e_1-e_2}{R^2}, \quad \beta^2 = \frac{e_2-e_3}{R^2}, \quad e_2 = -\frac{c}{12} \tag{42}$$

With the aid of relation (41) we find

$$g = -\frac{1}{R}[(e_1-e_2)(e_2-e_3)]^{1/2} \quad \text{for } \sigma = 0. \tag{43}$$

Then

$$T = \frac{1}{mR}\int_0^\sigma \frac{d\tau}{[(l^2+\tau^2)(\beta^2-\tau^2)]^{1/2}} = \frac{1}{m(e_1-e_3)^{1/2}}\left[F(r) - F\left(\sqrt{1-\frac{\sigma^2}{\beta^2}},\ r\right)\right], \tag{44}$$

where F(r) and F(u, r) are the complete and incomplete Legendre elliptic integrals of the first kind, respectively.

It follows from equation (44) that

$$\sigma = \frac{(e_2-e_3)^{1/2}}{R}\, cn\,[(e_1-e_3)^{1/2}(\omega - mT),\ r]. \tag{45}$$

The solution (45) cannot arise from solution (32) since, according to relations (40) and (42), we have

$$k^2 - \frac{c}{6} = \frac{c}{12} = e_2 \neq \frac{e_2(e_1-e_3)^{1/2} + e_3(e_1-e_2)^{1/2}}{(e_1-e_3)^{1/2} + (e_1-e_2)^{1/2}} .$$

Using relation (12) we can rewrite expression (33):

$$\theta = \frac{1}{C_\varepsilon}\left\{\exp(aT)\left(\int_0^\sigma T\exp(-aT)\,d\tau + \frac{9K}{a^2}\right) - \left[\int_0^\sigma T\,d\tau + 9K\left(\frac{T^2}{2} + \frac{T}{a} + \frac{1}{a^2}\right)\right]\right\}, \tag{46}$$

where C_ε is heat capacity under constant deformation, n = 0.

Since $T = \alpha\theta$, we then obtain the following expression for the coefficient of thermal expansion α:

$$\frac{1}{\alpha} = \frac{1}{C_\varepsilon}\left\{T^{-1}\exp(aT)\left(\int_0^\sigma T\exp(-aT)\,d\tau + \frac{9K}{a^2}\right) - \left[T^{-1}\int_0^\sigma T\,d\tau + 9K\left(\frac{T}{2} + \frac{1}{a} + \frac{T^{-1}}{a^2}\right)\right]\right\}. \tag{47}$$

From equation (47) we find that $\alpha \to 0$ as $a = m(EC_\varepsilon/\rho\lambda_0) \to 0$.

The quantity T in relations (46) and (47) is given by the expression (12) when $\lambda_3 > 0$ and by expression (44) when $\lambda_3 = 0$.

4. Condition of Existence and Nature of Solitary Waves. From relations (12) and (36) it follows that

$$\frac{d\sigma}{dT} = -mg > 0 \ \forall T. \tag{48}$$

Using relation (48) we can establish that σ attains a maximum value when $mT = \omega/2$; from relation (32) we then find

$$0 \leqslant \sigma \leqslant \frac{1}{R}\left[(e_2-e_3)\, H^{-\lambda_3\lambda_1}\frac{(e_1-e_3)^{1/2}}{(e_1-e_3)^{1/2}+(e_1-e_2)^{1/2}}\right]^{1/2}. \quad (49)$$

When $\lambda_3 = 0$, the quantity σ varies between the following limits:

$$0 \leqslant \sigma \leqslant \frac{1}{R}(e_2-e_3)^{1/2}. \quad (50)$$

For the cases $\lambda_3 > 0$ and $\lambda_3 = 0$ the quantity T lies in the following intervals, respectively:

$$0 \leqslant T \leqslant \frac{\omega}{2m}, \; 0 \leqslant T \leqslant \frac{\omega}{m}. \quad (51)$$

It follows from relations (8) and (11) that σ is a solution of the following equation:

$$\frac{\partial\sigma}{\partial x}+\frac{1}{c}\frac{\partial\sigma}{\partial t}=0. \quad (52)$$

It follows from this that

$$\sigma=\sigma(\xi), \text{ where } \xi=x-ct. \quad (53)$$

Relations (8), (11), (12), and (53) yield

$$\int_0^\sigma \frac{d\tau}{g}=x-ct=-mT, \quad (54)$$

so that relations (32) and (45) can be rewritten as

$$\sigma=\frac{1}{R}(e_2-e_3)^{1/2}H_1^{-\frac{1}{2}\lambda_3\lambda_1}\exp\left(-\frac{1}{2}\lambda_3\lambda_2\left(\frac{\omega}{2}+x-ct\right)\right)\times$$
$$\times\left\{\frac{(e_1-e_3)^{1/2}}{(e_1-e_3)^{1/2}+(e_1-e_2)^{1/2}}-sn^2\left[(e_1-e_3)^{1/2}\left(\frac{\omega}{2}+x-ct\right),\ r\right]\right\}, \quad (55)$$

$$\sigma=\frac{(e_2-e_3)^{1/2}}{R}cn[(e_1-e_3)^{1/2}(\omega+x-ct),\ r]. \quad (56)$$

Assume now that the source generating the solitary wave is located at the coordinate origin ($x = 0$). From relations (8), (36), (43), (55), and (56) we have:

$$\text{if } \lambda_3>0, \text{ then } \sigma=0, \ \frac{\partial\sigma}{\partial t}=+\infty \ \text{ for } x=0, \ t=0, \quad (57)$$

$$\text{if } \lambda_3=0, \text{ then } \sigma=0, \ \frac{\partial\sigma}{\partial t}=\frac{c}{R}[(e_1-e_2)(e_2-e_3)]^{1/2} \text{ for } x=0, \ t=0. \quad (58)$$

Only the initial condition (58) has an obvious physical meaning.

Thus, based on the above, we can conclude that the solitary waves (56), subject to the initial condition (58), can exist in a thermo-elastic medium if $\theta = \theta(\sigma)$, $\alpha = \alpha(\sigma)$ have the forms (46) and (47). These waves propagate along the positive direction of the x-axis with speed $c = 4k^2$. Their amplitude is given by

$$M=\frac{1}{R}(e_2-e_3)^{1/2}. \quad (59)$$

The width L of the solitary waves is given by the expression

$$L=2F(r). \quad (60)$$

A search of the literature shows that the conditions given for the appearance of solitary waves (56) are new.

NOTATION

$T = \alpha\theta$; $f^2 = E/\rho$, $K = 1/3(3\lambda + 2\mu)$; σ, stress; θ, temperature; α, coefficient of thermal expansion; K, E, λ, μ, elastic constants; κ, λ_0, C_ε, thermal constants; ρ, density of the medium.

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QUASISTATIC THERMOVISCOELASTIC FIELDS IN AN INFINITE BICOMPOSITE CYLINDRICALLY ISOTROPIC PLATE

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With the most general assumptions (within the framework of classical thermomechanics) nonsteady temperature fields and thermoviscoelastic fields of displacements and stresses induced by them are constructed in an infinite bicomponent cylindrically isotropic plate. Examples of Biot–Maxwell, Biot–Kelvin, and Maxwell–Kelvin plates are presented.

1. Nonsteady Temperature Fields. The problem of the structure of a nonsteady temperature field in an infinite bicomponent cylindrically isotropic (c. is.) plate leads mathematically to the construction of the solution of a separate system of B-parabolic equations bounded in the domain $D = \{(t, r);\ t \geq 0,\ r \in I_2^+ = 0), R)\cup(R, \infty)\}$ [1]

$$\frac{1}{a_j^2}\frac{\partial T_j}{\partial t} + \varkappa_j^2 T_j - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) T_j = f_j(t,\ r),\ j = 1,\ 2, \tag{1}$$

with the initial conditions

$$T_1|_{t=0} = g_1(r),\ r \in (0, R),\ T_2|_{t=0} = g_2(r),\ r \in (R, \infty), \tag{2}$$

and the conditions of nonideal thermal contact [2]

$$\left[\left(r_{12}\frac{\partial}{\partial t} + 1\right) T_1 - T_2\right]\Big|_{r=R} = 0,\ \left(\lambda_1 \frac{\partial T_1}{\partial r} - \lambda_2 \frac{\partial T_2}{\partial r}\right)\Big|_{r=R} = 0. \tag{3}$$

The solution of problem (1)-(3) can be constructed by the method of integral Fourier–Bessel transformation on the polar axis with one conjugate point [3]. We omit the mathematical operations and find that the nonsteady temperature field in the plate under consideration is described by the functions (on the assumption that $k_{12}^2 \equiv a_2^2\kappa_2^2 - a_1^2\kappa_1^2 \geq 0$)

$$T_j(t,\ r) = \frac{1}{a_1^2}\int_0^R H_{j1}(t,\ r,\ \rho)\, g_1(\rho)\,\rho d\rho + \frac{1}{a_2^2}\int_R^\infty H_{j2}(t,\ r,\ \rho)\, g_2(\rho)\,\rho d\rho +$$
$$+ \int_0^t \left[\int_0^R H_{j1}(t-\tau,\ r,\ \rho) f_1(\tau,\ \rho)\,\rho d\rho + \int_R^\infty H_{j2}(t-\tau,\ r,\ \rho) f_2(\tau,\ \rho)\,\rho d\rho\right] d\tau, \tag{4}$$
$$j = 1,\ 2.$$

In formulas (4) we introduced into consideration the influence functions

$$H_{11}(t,\ r,\ \rho) = \frac{4\lambda_1\lambda_2}{\pi^2 R^2} e^{-a_2^2\varkappa_2^2 t} \int_0^\infty J_0\left(\frac{b_1}{a_1} r\right) J_0\left(\frac{b_1}{a_1}\rho\right) e^{-\lambda^2 t}\frac{\lambda d\lambda}{\omega(\lambda)},$$

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